A linear combination of independent Normally distributed random variables has a Normal distribution.

If $X \sim \mathrm{N}(\mu_1, \sigma_1^{\,2})$ and $Y \sim \mathrm{N}(\mu_2, \sigma_2^{\,2})$
and $W = aX + bY$ then
$W \sim \mathrm{N}(a\mu_1 + b\mu_2, a^2\sigma_1^{\,2} + b^2\sigma_2^{\,2})$

Example

The random variable X is Normally distributed with mean 8 and standard deviation 3.

The independent random variable Y is Normally distributed with mean 15 and standard deviation 2.

Evaluate $P(Y - 2X < 0)$

Solution

$$X \sim \mathrm{N}(8, 3^2) \text{ and } Y \sim \mathrm{N}(15, 2^2)$$

$$\begin{aligned}
\mathrm{E}(Y - 2X) &= \mathrm{E}(Y) - 2\mathrm{E}(X) \\
&= 15 - 2 \times 8 = -1 \\
\mathrm{Var}(Y - 2X) &= \mathrm{Var}(Y) + (-2)^2\,\mathrm{Var}(X) \\
&= 4 + 2^2 \times 9 = 40
\end{aligned}$$

You can use your calculator to find $P(Y - 2X < 0)$ using the Normal CD function on your calculator. Use:

Lower: -9×10^{99}
Upper: 0
σ: $\sqrt{40}$
μ: -1

$\mathrm{P}(Y - 2X < 0) = 0{\cdot}56282$

Central Limit Theorem (CLT)

The **Central Limit Theorem** (CLT) states that the distribution of the mean of a large random sample from any distribution with known mean, μ, and variance, σ^2, is approximately Normally distributed.

$$\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

Example

$$X \sim \mathrm{N}(8, 3^2) \text{ and } Y \sim \mathrm{N}(15, 2^2)$$

Suppose Steve collects three observations of X and sums them, and Lucy collects two observations of Y and sums them. What is the probability that the difference between the two sums will be positive?

We want the distribution of $X_1 + X_2 + X_3 - Y_1 - Y_2$
Note: this is different to the distribution of $3X - 2Y$

So, $X_1 + X_2 + X_3 - Y_1 - Y_2$ has mean $8 + 8 + 8 - 15 - 15 = -6$
And variance $9 + 9 + 9 + 4 + 4 = 35$

Again, you can use your calculator to find P(difference between the two sums > 0), using the Normal CD function on your calculator, with:

Lower: 0
Upper: 9×10^{99}
σ: $\sqrt{35}$
μ: -6

P(difference between the two sums > 0) = 0·15525

The distribution of the mean of a random sample from a Normal distribution with known mean and variance is also Normal.

Example

A manufacturer produces two types of mobile phone cases: type A that weigh less than 32 g, and type B that weigh more than 32 g. The masses, X, in grams, of 8 mobile phone cases are collected from an unlabelled bag containing either all type A or all type B:

32·0, 31·3, 34·5, 32·8,
32·1, 32·5, 30·9, 31·2

You may assume that this is a random sample from a Normal population with standard deviation 0·5 g.

Find the probability that the unlabelled bag contains type A mobile phone cases.

Solution

$\bar{x} = 32{\cdot}1625$

Sample standard deviation $= \sqrt{\frac{0{\cdot}5^2}{8}}$

Again you can use your calculator to find the probability, with:

Lower: -9×10^{99}
Upper: 32
σ: 0·17677669
μ: 32·1625

The probability that the unlabelled bag contains type A mobile phone cases is 0·17899.