A level Further Maths

6.3 Moments and Centre of Mass

Terminology and Modelling

Centre of mass – The point through which the mass of an object is concentrated.

Moment – A measure of the turning effect of a force on a rigid body.

Coplanar – Acting in the same plane.

Lamina – A 2D object whose thickness can be ignored.

Uniform – Mass is distributed evenly.

Density – Often denoted by ρ.
In 2D, a measure of mass per unit area, units kg/m^2
In 3D, a measure of mass per unit volume, units kg/m^3

2D Problems

From page 8 of the formula booklet. →

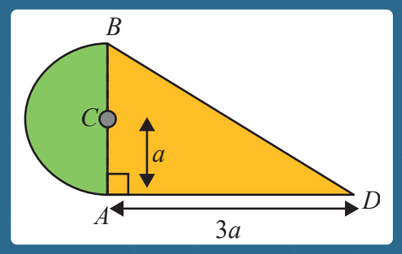
Centres of Mass of Uniform Bodies

Triangular lamina: $\frac{2}{3}$ along median from vertex

Semi circle: $\frac{4r}{3\pi}$ from straight edge along axis of symmetry

Quarter circle: $\bar{x} = \frac{4r}{3\pi}$ $\bar{y} = \frac{4r}{3\pi}$ from vertex

ACB is the diameter of a semi-circular lamina with centre C. A right-angled triangular lamina ADB is added as well as a particle at C to form a uniform lamina, as shown below. The particle at C has a mass equal to twice that of the semi-circle.



Shape	Area/mass	Distance from AB	Distance from AD
	$\frac{1}{2}\pi a^2\rho$	$(-)\frac{4a}{3\pi}$	a
	$3a^2\rho$	a	$\frac{2a}{3}$
	$\pi a^2\rho$	0	a
	$a^2\rho\left(\frac{3}{2}\pi + 3\right)$	$\bar{x}$	$\bar{y}$

Moments about AB

$$a^2\rho\left(\frac{3}{2}\pi + 3\right)\bar{x} = \left(\frac{1}{2}\pi a^2\rho\right)\left(-\frac{4a}{3\pi}\right) + (3a^2\rho)(a)$$

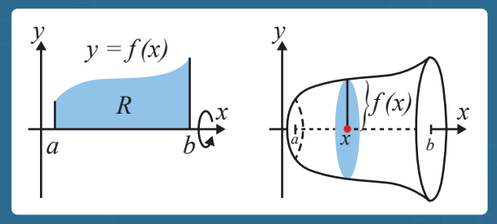
$$\bar{x} = \frac{7}{3\left(\frac{3}{2}\pi + 3\right)}a$$

Moments about AD

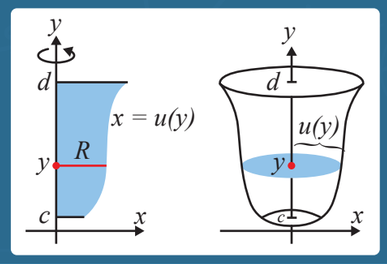
$$a^2\rho\left(\frac{3}{2}\pi + 3\right)\bar{y} = \left(\frac{1}{2}\pi a^2\rho\right)(a) + (3a^2\rho)\left(\frac{2a}{3}\right) + (\pi a^2\rho)(a)$$

$$\bar{y} = \frac{\left(\frac{3}{2}\pi + 2\right)}{\left(\frac{3}{2}\pi + 3\right)}a$$

Volume of revolution



$V\bar{x} = \pi\int_a^b xy^2\,dx$ where $V = \pi\int_a^b y^2\,dx$

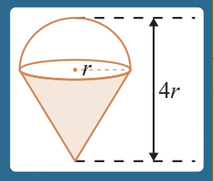


$V\bar{y} = \pi\int_c^d yx^2\,dy$ where $V = \pi\int_c^d x^2\,dy$

3D Problems

3D Composite body

A composite solid body consists of a uniform cone capped by a uniform hemisphere. The total height of the solid is $4r$ cm, where r represents the common radius. The ratio of the density of the hemisphere to that of the cone is 3:1.



Shape	Mass	Distance from Vertex
	$\frac{2}{3}\pi r^3 \times 3\rho$ $(= 2\pi r^3\rho)$	$3r + \frac{3r}{8}$ $\left(= \frac{27}{8}r\right)$
	$\frac{1}{3}\pi r^2 \times 3r \times \rho$ $(= \pi r^3\rho)$	$\frac{3}{4}(3r) = \frac{9}{4}r$
	$\pi r^3\rho + 2\pi r^3\rho$ $(= 3\pi r^3\rho)$	$\bar{h}$

$\bar{h}$ is the distance of the centre of mass vertically above the vertex of the cone.

The following results are not provided:

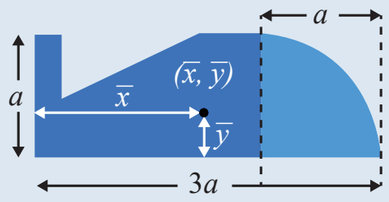
Diagram	Body	Volume/ Curved Surface Area	Height of COM above base
	Solid sphere, radius r	$\frac{4}{3}\pi r^3$	r
	Solid hemisphere, radius r	$\frac{2}{3}\pi r^3$	$\frac{3}{8}r$
	Hollow hemisphere, radius r	$2\pi r^2$	$\frac{1}{2}r$
	Solid cone or pyramid, height h, radius r	$\frac{1}{3}\pi r^2 h$	$\frac{1}{4}h$
	Hollow cone or pyramid, height h, radius r	πrl *(l is sloping height)*	$\frac{1}{3}h$

Moments about vertical through vertex

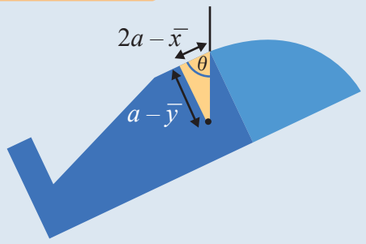
$$3\pi r^3\rho \times \bar{h} = \pi r^3\rho \times \frac{9}{4}r + 2\pi r^3\rho \times \frac{27}{8}r$$

$$\bar{h} = 3r$$

Suspension from a fixed point

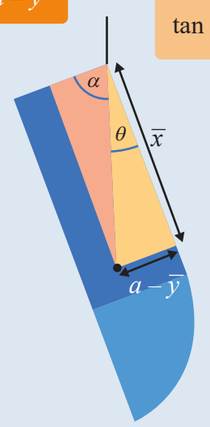


$\tan\theta = \frac{a - \bar{y}}{2a - \bar{x}}$



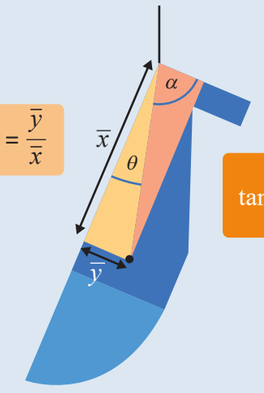
$\tan\alpha = \frac{\bar{x}}{a - \bar{y}}$

$\tan\theta = \frac{a - \bar{y}}{\bar{x}}$



$\tan\theta = \frac{\bar{y}}{\bar{x}}$

$\tan\alpha = \frac{\bar{x}}{\bar{y}}$



$\tan\theta = \frac{\bar{y}}{3a - \bar{x}}$